Question One (8 marks)

a) What is the common difference of the arithmetic series $2+\frac{1}{2}-1+...$? **1**

b) Evaluate $\sum_{k=3}^{6} k^2$ **2**

c) In a geometric series the 3rd term is 16 and the 7th term is 256. Find the common ratio of the series. **3**

d) Find $f''(x)$ if $f(x)=2x^4$ **2**

Question Two (8 marks)

a) Simplify $\cos 15° \sin 30° - \sin 15° \cos 30°$ (Do not evaluate) **2**

b) Prove the following identity: $\frac{2\sin^2 x - 1}{\sin x + \cos x} \equiv \sin x - \cos x$ **3**

c) Solve $3\cos\theta + 4\sin\theta = -3$ for $0° \le \theta \le 360°$ using the substitution $t = \tan\frac{\theta}{2}$. **3**

Question Three (8 marks)

a) Prove the following result using Mathematical Induction:
$1^3 + 2^3 + 3^3 + ... + n^3 = \frac{1}{4}n^2(n+1)^2$ where *n* is a positive integer **4**

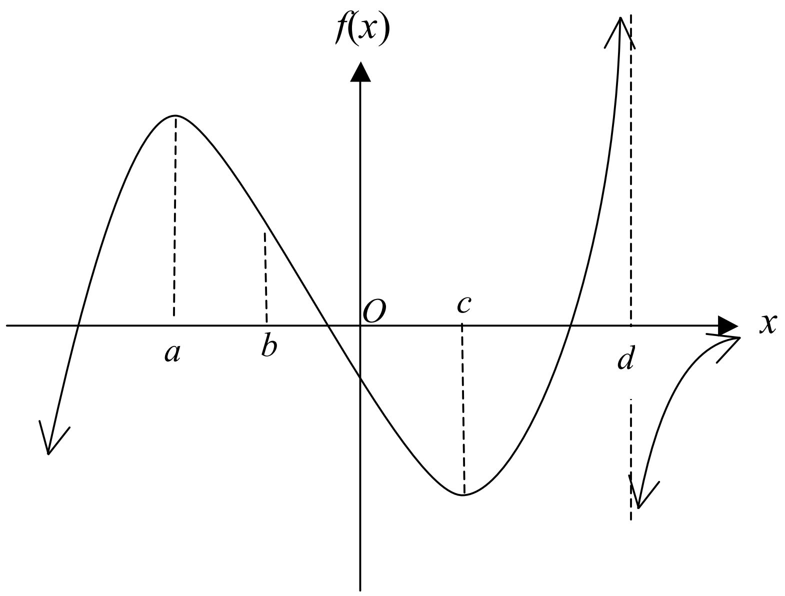
b) *L* is the fixed point $(-1,4)$ and *P* is a variable point on the parabola $x^2 = 8y$. *M* is the midpoint of *LP*. Find the equation of the locus of *M*. **4**

Question Four (9 marks)

a) Show that $\dfrac{x^2+9}{x-4} = x+4+\dfrac{25}{x-4}$ **1**

b) Sketch $y = \dfrac{x^2+9}{x-4}$ showing all turning points, intercepts and asymptotes.
(Do not find any points of inflexion) **6**

c) Consider this graph, which has a point of inflexion at $x = b$.



For what values of *x* is $f'(x) > 0$? **2**

Question Five (8 marks)

a) Consider the series: $(x+3)-(x+3)^2+(x+3)^3-(x+3)^4+...$

i. For what values of *x* does this series have a limiting sum? **2**

ii. Find the limiting sum in terms of *x*. **2**

iii. Hence show that the limiting sum of this series cannot be greater than 1 **1**

b) The formula for the *n*th term of a series is $T_n = (n+2)^3 - (n+1)^3$.

i. Evaluate S_5 for this series. **2**

ii. Find the sum of the first 100 terms of this series. **1**

Question Six (9 marks)

a) Show by Mathematical Induction that $5n - 7 \le 2^n$ where n is an integer greater than 2. **3**

b) $f(x)$ is the function $y = x^3 - ax^2 + b$, where a and b are positive.

i. Show that graph of $y = f(x)$ has a non-horizontal point of inflexion at $x = \frac{a}{3}$. **3**

ii. Given that $x_0 \le \frac{2a}{3}$, show that $f(x_0) \le b$. **3**

Question One (8 marks)

a) -1.5

b) $3^2 + 4^2 + 5^2 + 6^2 = 86$

c)

$ar^6 = 256 \quad -(1)$

$ar^2 = 16 \quad -(2)$

$(1) \div (2)$

$r^4 = 16$

$r = \pm 2$

d)

$f'(x) = 8x^3$

$f''(x) = 24x^2$

Question Two (8 marks)

a)

$\sin(30 - 15)$

$= \sin 15°$

b)

$$\begin{aligned} lhs &= \frac{2\sin^2 x - 1}{\sin x + \cos x} \\ &= \frac{2\sin^2 x - (\sin^2 x + \cos^2 x)}{\sin x + \cos x} \\ &= \frac{\sin^2 x - \cos^2 x}{\sin x + \cos x} \\ &= \frac{(\sin x + \cos x)(\sin x - \cos x)}{\sin x + \cos x} \\ &= \sin x - \cos x \\ &= rhs \end{aligned}$$

c)

$$\frac{3 \times (1 - t^2)}{1 + t^2} + \frac{4 \times 2t}{1 + t^2} = -3$$

$$\begin{aligned} -3t^2 + 8t + 3 &= -3(1 + t^2) \\ &= -3 - 3t^2 \\ 8t + 6 &= 0 \\ t &= -\frac{3}{4} \\ \tan\frac{\theta}{2} &= -\frac{3}{4} \\ \frac{\theta}{2} &= 143.1301..., \ 0 \le \frac{\theta}{2} \le 180 \\ \theta &= 286°16' \end{aligned}$$

Check $\theta = 180$:

$$\begin{aligned} lhs &= 3\cos(180°) + 4\sin(180°) \\ &= 3 \times -1 + 4 \times 0 \\ &= -3 \\ &= rhs \end{aligned}$$

So $\theta = 180, \ 286°16'$

Question Three (7 marks)

a)

When n=1:

$$\begin{aligned} lhs &= 1^3 \\ &= 1 \\ rhs &= \left(\frac{1}{4} \times 1^2 \times (1+1)^2\right) \\ &= 1 \\ &= lhs \end{aligned}$$

So the statement is true when $n = 1$

Let k be a value of *n* for which the statement is true

Then

$$1^3 + 2^3 + 3^3 + ... + k^3 = \frac{1}{4}k^2(k+1)^2$$

We want to show that

$$1^3 + 2^3 + 3^3 + ... + k^3 + (k+1)^3 = \frac{1}{4}(k+1)^2(k+2)^2$$

$$\begin{aligned} lhs &= \frac{1}{4}k^2(k+1)^2 + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\ &= (k+1)^2\left[\frac{k^2}{4} + k + 1\right] \\ &= (k+1)^2\left(\frac{k^2 + 4k + 4}{4}\right) \\ &= (k+1)^2\left(\frac{(k+2)^2}{4}\right) \\ &= \frac{1}{4}(k+1)^2(k+2)^2 \\ &= rhs \end{aligned}$$

So if the statement is true when $n = k$ then it is also true when $n = k + 1$.

Since the statement is true when $n = 1$ it must also be true for $n = 2$, and since it is true for $n = 2$, it must also be true for $n = 3$ and so on for all positive integers n.

b)

$4a = 8$

$a = 2$

So P has coordinates $(2ap, ap^2)$ i.e. $(4p, 2p^2)$

Let (X, Y) be the coordinates of M.
Using the midpoint formula:

$X = \dfrac{-1+4p}{2}$ $-(1)$ and

$Y = \dfrac{4+2p^2}{2}$

$= 2 + p^2$ $-(2)$

From (1) $p = \dfrac{2X+1}{4}$

Into (2) :

$Y = 2 + \left(\dfrac{2X+1}{4}\right)^2$

So the locus is the parabola

$\left(x + \dfrac{1}{2}\right)^2 = 4(y-2)$

Question Four (6 marks)

a)
i)

$\dfrac{x^2+9}{x-4} = \dfrac{x(x-4)+4(x-4)+25}{x-4}$

$= x + 4 + \dfrac{25}{x-4}$

ii)
The vertical asymptote occurs at $x - 4 = 0$ i.e. $x = 4$.

As $x \longrightarrow \infty$, $\dfrac{25}{x-4} \longrightarrow 0$ so

$y \longrightarrow x + 4$

So $y = x + 4$ is an oblique asymptote

Since $x^2 + 9$ cannot equal zero there are no x-intercepts.

The y-intercept is $y = \dfrac{0^2+9}{0-4}$ i.e. $y = -\dfrac{9}{4}$

$y' = \dfrac{(x-4)(2x) - (x^2+9)(1)}{(x-4)^2}$

Stationary points will occur when

$(x-4)(2x) - (x^2+9)(1) = 0$

$x^2 - 8x - 9 = 0$

$(x-4)^2 = 9 + 16$

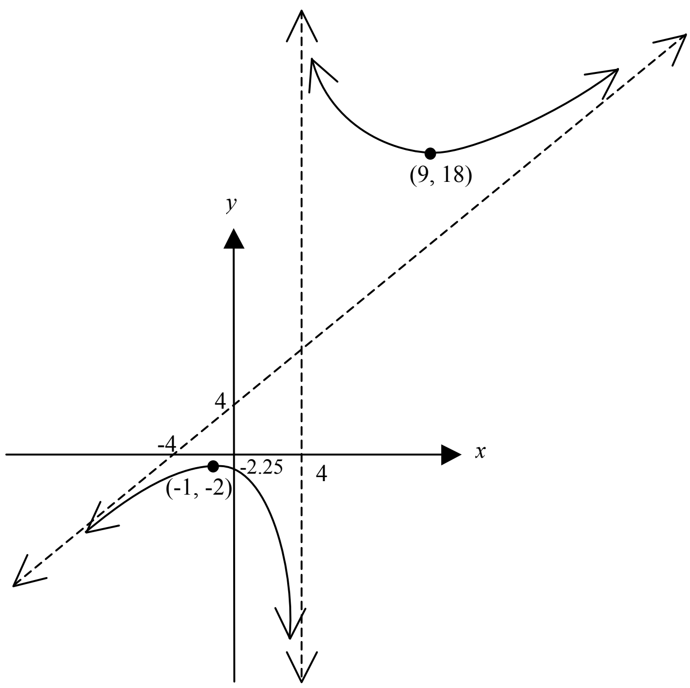
$x = 4 \pm \sqrt{25}$

$x = -1, 9$

x	-2	-1	0	9	10
y'	$\frac{11}{36}$	0	$-\frac{9}{16}$	0	$\frac{11}{36}$

There is a maximum turning point at $x = -1$ and a minimum turning point at $x = 9$.

When $x = -1$ $y = -2$ and when $x = 9$ $y = 18$



b) $x < a,\ c < x < d,\ x > d$

Question Five (6 marks)

a)

i)

$$-1 < -(x+3) < 1$$
$$-1 < x+3 < 1$$
$$-4 < x < -2$$

ii)

$$\frac{a}{1-r} = \frac{x+3}{1--(x+3)}$$
$$= \frac{x+3}{x+4}$$

iii)
now, for all x

$x+3 < x+4$

We can divide both sides by $x+4$ since from part (i) $x+4>0$

$$\frac{x+3}{x+4} < 1$$

So the limiting sum cannot be greater than one.

<u>Alternative proof:</u>
Suppose the limiting sum were greater than one.

Then $\frac{x+3}{x+4} > 1$.

We can multiply both sides by $x+4$ since from part (i) $x+4>0$

$x+3 > x+4$
But this would mean that $3>4$ which is false. This means that our original assumption must be false so the limiting sum cannot be greater than one.

b)

$$S_5 = (1+2)^3 - (1+1)^3 + (2+2)^3 - (2+1)^3$$
$$+ (3+2)^3 - (3+1)^3 + (4+2)^3 - (4+1)^3$$
$$+ (5+2)^3 - (5+1)^3.$$
$$= 3^3 - 2^3 + 4^3 - 3^3 + 5^3 - 4^3 + 6^3. - 5^3$$
$$+ 7^3 - 6^3$$
$$= 335$$

$$S_{100} = (1+2)^3 - (1+1)^3 + (2+2)^3 - (2+1)^3$$
$$+ (3+2)^3 - (3+1)^3 ... + (100+2)^3$$
$$- (100+1)^3$$
$$= 3^3 - 2^3 + 4^3 - 3^3 + 5^3$$
$$- 4^3 + ... + 102^3 - 101^3$$

Almost every term in the sum cancels so

$$S_{100} = 102^3 - 2^3$$
$$= 1061200$$

Question Six (7 marks)

a)

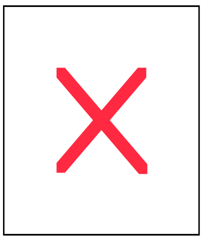
Let n=3
$lhs = 5(3) - 7$
$= 8$
$rhs = 2^3$
$= 8$
$\geq lhs$

Let k be a value of n for which the statement is true
i.e. $5k - 7 \leq 2^n$

we want to show that $5(k+1) - 7 \leq 2^{k+1}$

note that since $k \geq 3$,

$$2^k \geq 2^3$$
$$= 8$$



So if the statement is true when $n = k$ then it is also true when $n = k+1$.

Since the statement is true when $n = 3$ it must also be true for $n = 4$, and since it is true for $n = 4$, it must also be true for $n = 5$ and so on for all positive integers n.

b)

i.

$f(x) = x^3 - ax^2 + b$
$f'(x) = 3x^2 - 2ax$
$f''(x) = 6x - 2a$

When $x = \frac{a}{3}$,

$$f''(x) = 6\left(\frac{a}{3}\right) - 2a$$
$$= 0$$

$$f'(x) = 3\left(\frac{a}{3}\right)^2 - 2a\left(\frac{a}{3}\right)$$
$$= -\frac{a^2}{3}$$
< 0, since $a > 0$

When $x < \frac{a}{3}$

$$6x < \frac{6a}{3}$$
$$so\ \ 6x - 2a < \frac{6a}{3} - 2a$$
$$= 0$$

i.e. $f''(x) < 0$

When 

$$6x > \frac{6a}{3}$$
$$so\ \ 6x - 2a > \frac{6a}{3} - 2a$$
$$= 0$$

i.e. $f''(x) > 0$

So when $x = \frac{a}{3}$, the second derivative equals zero and changes signs and the first derivative is not zero i.e. there is a non-horizontal point of inflexion at $x = \frac{a}{3}$.

ii) Note that since $f(x)$ is a polynomial, the graph of $y = f(x)$ is continuous.

Stationary points occur when $f'(x) = 0$

$3x^2 - 2ax = 0$
$x(3x - 2a) = 0$
$x = 0, \frac{2a}{3}$

When $x = 0$

$f''(x) = 6(0) - 2a$
$= -2a$
< 0 since $a > 0$

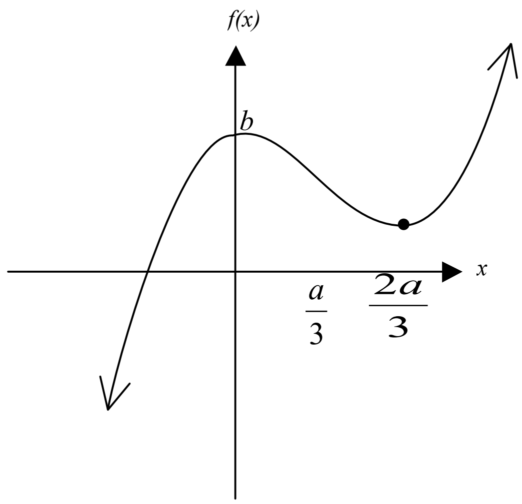
So there is a maximum turning point at (0,b)

When $x = \frac{2a}{3}$,

$f''(x) = 6(\frac{2a}{3}) - 2a$
$= 2a$
> 0 since $a > 0$

So there is minimum turning point at $x = \frac{2a}{3}$

The location of these points depends on *a* and b but we can conclude that the *shape* of the graph is:



So when $x_0 \le \frac{2a}{3}$, show that $f(x_0) \le b$.